

# Fourier spectral methods for some linear stochastic space-fractional partial differential equations

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## Abstract

Fourier spectral methods for solving some linear stochastic space-fractional partial differential equations perturbed by space-time white noises in one-dimensional case are introduced and analyzed. The space-fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. We approximate the space-time white noise by using piecewise constant functions and obtain the approximated stochastic space-fractional partial differential equations. The approximated stochastic space-fractional partial differential equations are then solved by using Fourier spectral methods. Error estimates in  $L^2$ - norm are obtained. Numerical examples are given.

*Key words:*

Space-fractional partial differential equations, stochastic partial differential equations, Fourier spectral method, error estimates

*AMS Subject Classification:* 65M12; 65M06; 65M70; 35S10

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## 1. Introduction

In this paper we will consider Fourier spectral method for solving the following linear stochastic space-fractional partial differential equation, with  $1/2 < \alpha \leq 1$ ,

$$\frac{\partial u(t, x)}{\partial t} + (-\Delta)^\alpha u(t, x) = \frac{\partial^2 W(t, x)}{\partial t \partial x}, \quad 0 < t < T, \quad 0 < x < 1, \quad (1)$$

$$u(t, 0) = u(t, 1) = 0, \quad 0 < t < T, \quad (2)$$

$$u(0, x) = u_0(x), \quad 0 < x < 1, \quad (3)$$

where  $(-\Delta)^\alpha$  is the fractional Laplacian and  $\frac{\partial^2 W(t, x)}{\partial t \partial x}$  is the mixed second order derivative of the Brownian sheet [40]. It is well-known that the Laplacian  $-\Delta$  has eigenpairs  $(\lambda_j, e_j)$  with  $\lambda_j = j^2\pi^2$ ,  $e_j = \sqrt{2} \sin j\pi x$ ,  $j = 1, 2, 3, \dots$  subject to the homogeneous Dirichlet boundary conditions on  $(0, 1)$ , i.e.,  $e_j(0) = e_j(1) = 0$  and

$$-\Delta e_j = \lambda_j e_j, \quad j = 1, 2, 3, \dots$$

Let  $H = L^2(0, 1)$  with inner product  $(\cdot, \cdot)$  and norm  $\|\cdot\|$ . For any  $r \in \mathbb{R}$ , we denote

$$H_0^r := \left\{ v : v = \sum_{j=1}^{\infty} (v, e_j) e_j, \quad \text{where } \sum_{j=1}^{\infty} \lambda_j^r (v, e_j)^2 < \infty \right\},$$

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Preprint submitted to Elsevier

October 6, 2016

with norm

$$|v|_r = \left( \sum_{j=1}^{\infty} \lambda_j^r(v, e_j)^2 \right)^{1/2}.$$

Then, for any  $v \in H_0^{2\alpha}(0, 1)$ ,  $1/2 < \alpha \leq 1$ , we have

$$(-\Delta)^\alpha v = \sum_{j=1}^{\infty} (v, e_j) \lambda_j^\alpha e_j. \quad (4)$$

Space-fractional partial differential equations are widely used to model complex phenomena, for example, in quasi geostrophic flow, the fast rotating fluids, the dynamic of the frontogenesis in meteorology, the diffusions in fractal or disordered medium, the pollution problems, the mathematical finance and the transport problems, see, e.g., [5], [9], [24], [42], [4]. Therefore it is natural to consider space-fractional partial differential equations perturbed by some noises. In [28], the authors considered the fluctuating interfaces, the evolution of the height  $h(t, x)$  of the interface is usually written in Langevin form, with  $1/2 < \alpha \leq 1$ ,

$$\frac{\partial h(t, x)}{\partial t} = (-\Delta)^\alpha h(t, x) + \eta(t, x), \quad (5)$$

where  $\eta(t, x)$  represents the space-time white noise.

The existence, uniqueness and regularities of the solutions to stochastic space-fractional partial differential equations have been extensively studied, see, e.g., [5], [9], [10], [33]. But the numerical methods for solving space-fractional stochastic partial differential equations are quite restricted. Debbi and Dozzi [11] introduced a discretization to the fractional Laplacian and used it to elaborate an approximation scheme for fractional heat equation perturbed by a multiplicative cylindrical white noise. We have not seen any other research papers of numerical methods for solving stochastic space-fractional partial differential equations. In this work, we will use the ideas in [1] to consider the numerical methods for solving stochastic space-fractional partial differential equations, see also [14], [22], [8], [23]. We first approximate the space-time white noise by using piecewise constant functions and obtain the approximate solution  $\hat{u}(t)$  of the exact solution  $u(t)$  and then prove the error estimates in  $L^2$ -norm for  $u(t) - \hat{u}(t)$ .

For the deterministic space-fractional partial differential equations, many numerical methods are available in literature. There are two ways to define the fractional Laplacian. One way of defining  $(-\Delta)^\alpha v$ ,  $1/2 < \alpha \leq 1$  is by using the eigenvalues and eigenfunctions of the Laplacian  $-\Delta$  subject to the boundary conditions as in (4). Another way of defining  $(-\Delta)^\alpha v$ ,  $1/2 < \alpha \leq 1$  is by using the integral for the function  $\tilde{v}$ , where  $\tilde{v}$  is defined on the whole real line  $\mathbb{R}$  and is the extension function of  $v$

$$\tilde{v}(x) = \begin{cases} v(x), & 0 < x < 1, \\ 0, & x \notin (0, 1). \end{cases}$$

More precisely, for  $\tilde{v}(x)$ ,  $x \in \mathbb{R}$ , we define

$$(-\Delta)^\alpha \tilde{v}(x) = C_\alpha \int_{\mathbb{R} \setminus \{0\}} \frac{2\tilde{v}(x) + \tilde{v}(x+y) - \tilde{v}(x-y)}{|y|^{1+2\alpha}} dy, \quad x \in \mathbb{R},$$

where  $C_\alpha$  is a positive constant depending on  $\alpha$ . We then define, [34]

$$(-\Delta)^\alpha \tilde{v}(x) = \mathcal{F}^{-1} \left( |\xi|^{2\alpha} (\mathcal{F}(\tilde{v}))(\xi) \right), \quad x \in \mathbb{R},$$

where  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  denote the Fourier and inverse Fourier transforms, respectively. For  $v(x)$ ,  $x \in (0, 1)$ , we define the fractional Laplacian by

$$(-\Delta)^\alpha v(x) = (-\Delta)^\alpha \tilde{v}(x),$$

It is easy to show that for some suitable functions  $w(x)$ ,  $x \in \mathbb{R}$ , [41]

$$(-\Delta)^\alpha w(x) = \mathcal{F}^{-1} \left( |\xi|^{2\alpha} w(\xi) \right) = \frac{1}{2 \cos(\pi\alpha)} \left( {}^R D_x^{2\alpha} w(x) + {}^R D_\infty^{2\alpha} w(x) \right),$$

where  ${}^R_{-\infty}D_x^{2\alpha}w(x)$  and  ${}^R_xD_{\infty}^{2\alpha}w(x)$ ,  $1 < \beta < 2$  are called Riemann-Liouville fractional derivatives defined by, with  $1 < \beta < 2$ ,

$$\begin{aligned} {}^R_{-\infty}D_x^{\beta}w(x) &= \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_{-\infty}^x (x-y)^{1-\beta} w(y) dy, \\ {}^R_xD_{\infty}^{\beta}w(x) &= \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_x^{\infty} (y-x)^{1-\beta} w(y) dy, \end{aligned}$$

Hence for the function  $v(x)$  defined on the bounded interval  $(0, 1)$ , we have

$$(-\Delta)^{\alpha}v(x) = \frac{1}{2\cos(\pi\alpha)} \left( {}^R_0D_x^{2\alpha}v(x) + {}^R_xD_1^{2\alpha}v(x) \right), \quad x \in (0, 1), \quad (6)$$

which is also called Riesz fractional derivative.

We remark that the definitions (4) and (6) are not equivalent [34]. For the deterministic space-fractional partial differential equations where the space-fractional derivative is defined by (6), or the Riemann-Liouville space-fractional derivative, or the Caputo space-fractional derivative, many numerical methods are available, e.g., finite difference methods [2],[3], [20]-[21], [29]-[31], [35]-[39],[41], finite element methods [12],[13]-[19], [32], [43], [44], and spectral methods [25]-[26]. For the deterministic space-fractional partial differential equations where the space-fractional derivative is defined by (4), some numerical methods are also available, see, e.g., matrix transfer method (MTT) [20], [21], [7], Fourier spectral method [6]. In this work, we will use Fourier spectral method to solve the approximated stochastic space-fractional partial differential equations. The main advantage of this approach is that it gives a full diagonal representation of the fractional operator, being able to achieve spectral convergence regardless of the fractional power in the problem. Let  $0 = x_0 < x_1 < x_2 < \dots < x_J = 1$  be the space partition of  $(0, 1)$  and  $h$  the space stepsize. Let  $0 = t_0 < t_1 < t_2 < \dots < t_N = T$  be the time partition of  $(0, T)$  and  $k$  the time stepsize. To find the approximate solution of (1)-(3), we first approximate the space-time white noise  $\frac{\partial^2 W(t, x)}{\partial t \partial x}$  by using a piecewise constant function  $\frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}$  defined by, with  $n = 1, 2, 3, \dots, N, j = 1, 2, \dots, J$ , [1]

$$\frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x} := \frac{\eta_{nj}}{\sqrt{kh}}, \quad t_{n-1} \leq t \leq t_n, \quad x_{j-1} \leq x \leq x_j, \quad (7)$$

where  $\eta_{nj} \in \mathcal{N}(0, 1)$  is an independently and identically distributed random variable and

$$\eta_{nj} = \frac{1}{\sqrt{kh}} \int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} dW(t, x).$$

Hence

$$\frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x} = \frac{1}{kh} \int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} dW(t, x), \quad \text{on } [t_{n-1}, t_n] \times [x_{j-1}, x_j]. \quad (8)$$

We also note that, [1]

$$\int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} d\hat{W}(t, x) = \int_{t_{n-1}}^{t_n} \int_{x_{j-1}}^{x_j} \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x} dx dt.$$

The solution  $u(t, x)$  of (1)-(3) can then be approximated by  $\hat{u}(t, x)$  where

$$\frac{\partial \hat{u}(t, x)}{\partial t} + (-\Delta)^{\alpha} \hat{u}(t, x) = \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}, \quad 0 < t < T, \quad 0 < x < 1, \quad (9)$$

$$\hat{u}(t, 0) = \hat{u}(t, 1) = 0, \quad 0 < t < T, \quad (10)$$

$$\hat{u}(0, x) = u_0(x), \quad 0 < x < 1. \quad (11)$$

Note that  $\frac{\partial^2 \hat{W}(t,x)}{\partial t \partial x}$  now is a function in  $L^2((0,T) \times (0,1))$  and therefore we can solve (9)-(11) by using any numerical methods for solving deterministic space-fractional partial differential equations. In Theorem 2.2, we prove that, with  $1/2 < \alpha \leq 1$ ,

$$\mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}(t,x))^2 dx dt \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}). \quad (12)$$

Let us now introduce Fourier spectral method for solving (9)-(11). Let  $J$  be a positive integer and denote

$$S_J = \text{span}\{e_1, e_2, \dots, e_J\}.$$

Define by  $P_J : H \rightarrow S_J$  the projection from  $H$  to  $S_J$ ,

$$P_J v = \sum_{j=1}^J (v, e_j) e_j. \quad (13)$$

The Fourier spectral method for solving (9)-(11) is to find  $\hat{u}_J(t) \in S_J$  such that

$$\frac{\partial \hat{u}_J(t,x)}{\partial t} + (-\Delta)^\alpha \hat{u}_J(t,x) = P_J \frac{\partial^2 \hat{W}(t,x)}{\partial t \partial x}, \quad 0 < t < T, \quad 0 < x < 1, \quad (14)$$

$$\hat{u}_J(t,0) = \hat{u}_J(t,1) = 0, \quad 0 < t < T, \quad (15)$$

$$\hat{u}_J(0,x) = P_J u_0(x), \quad 0 < x < 1, \quad (16)$$

In Theorem 3.1, we prove that, with  $1/2 < \alpha \leq 1$ ,

$$\|\hat{u}(t) - \hat{u}_J(t)\| \leq C\|u_0 - P_J u_0\| + C \frac{1}{(J+1)^\alpha} \left( \int_0^t \|\hat{f}(s)\|^2 ds \right)^{1/2}. \quad (17)$$

Combining Theorem 2.2 with Theorem 3.1, we have

$$\begin{aligned} \mathbb{E} \int_0^T \int_0^1 (u(t,x) - \hat{u}_J(t,x))^2 dx dt &\leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}) + C\|u_0 - P_J u_0\|^2 \\ &\quad + C \frac{1}{(J+1)^{2\alpha}} \left( k^{-1-\frac{1}{2\alpha}} + k^{-1} h^{-1} \right). \end{aligned}$$

The paper is organized as follows. In Section 2, we consider the approximation of space-time white noise. In Section 3, we consider the Fourier spectral methods for deterministic space-fractional partial differential equations and the error estimates are proved. In Section 4, we consider a numerical example.

## 2. Approximate white noise and regularity

Consider the stochastic space-fractional partial differential equation

$$\frac{\partial u(t,x)}{\partial t} + (-\Delta)^\alpha u(t,x) = f(t,x), \quad 0 < t < T, \quad 0 < x < 1, \quad (18)$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \quad (19)$$

$$u(0,x) = u_0(x), \quad 0 < x < 1, \quad (20)$$

where  $f(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}$  denotes the mixed second order derivative of the Brownian sheet. [1] There is no strong solution of (18)-(20) since  $f(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x} \notin L^2((0,T) \times (0,1))$ .

The mild solution of (18)-(20) has the following form, see, e.g., [10], [33],

$$u(t, x) = \int_0^1 G_\alpha(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G_\alpha(t-s, x, y) dW(s, y), \quad (21)$$

where

$$G_\alpha(t, x, y) = \sum_{j=1}^{\infty} e^{-\lambda_j^\alpha t} e_j(x) e_j(y),$$

and the stochastic integral  $\int_0^t \int_0^1 G_\alpha(t-s, x, y) dW(s, y)$  is well-defined.

We have the following existence and uniqueness theorem, see, e.g., [10], [33]

**Theorem 2.1.** *Let  $1/2 < \alpha \leq 1$  and  $\beta > 0$ . Let  $u_0$  be a  $H_0^\beta(0, 1)$ -valued  $\mathcal{F}_0$ -measurable function such that*

$$\mathbb{E} \|u_0\|_{H_0^\beta(0,1)}^p < \infty,$$

*for some  $p > \frac{4\alpha}{2\alpha-1}$ . Then (18)-(20) has a unique mild solution  $u$  such that, for any  $0 \leq \theta < \min\{\frac{2\alpha-1}{2} - \frac{2\alpha}{p}, \beta\}$ ,*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u(t)\|_{H_0^\theta(0,1)}^p < \infty.$$

We approximate the solution  $u(t, x)$  of (18)-(20) by  $\hat{u}(t, x)$  where

$$\frac{\partial \hat{u}(t, x)}{\partial t} + (-\Delta)^\alpha \hat{u}(t, x) = \hat{f}(t, x), \quad 0 < t < T, \quad 0 < x < 1, \quad (22)$$

$$\hat{u}(t, 0) = \hat{u}(t, 1) = 0, \quad 0 < t < T, \quad (23)$$

$$\hat{u}(0, x) = u_0(x), \quad 0 < x < 1. \quad (24)$$

Here  $\hat{f}(t, x) = \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}$  is defined by (7). The solution of (22)-(24) has the form of, see, e.g., [1]

$$\hat{u}(t, x) = \int_0^1 G_\alpha(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G_\alpha(t-s, x, y) d\hat{W}(s, y), \quad (25)$$

**Theorem 2.2.** *Let  $u$  and  $\hat{u}$  be the solutions of (18)-(20) and (22)-(24), respectively. Assume that  $u_0 \in H$ . We have, with  $1/2 < \alpha \leq 1$ ,*

$$\mathbb{E} \int_0^T \int_0^1 (u(t, x) - \hat{u}(t, x))^2 dx dt \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}). \quad (26)$$

**Proof:** See the Appendix. ■

**Remark 2.1.** *When  $\alpha = 1$ , we obtain the same estimates as in [1] and [14], i.e.,*

$$\mathbb{E} \int_0^T \int_0^1 (u(t, x) - \hat{u}(t, x))^2 dx dt \leq C(k^{\frac{1}{2}} + h^2 k^{-\frac{1}{2}}).$$

**Theorem 2.3.** *Let  $\hat{u}$  be the solution of (22)-(24). We have, with  $1/2 < \alpha \leq 1$ ,*

$$\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \hat{u}_t^2(t, x) dx dt \leq C(k^{-\frac{1}{2\alpha}} + h^{-1}), \quad j \geq 0, \quad (27)$$

and

$$\mathbb{E} \int_0^1 \left| (-\Delta)^\alpha \hat{u}(t, x) \right|^2 dx \leq C(k^{-1-\frac{1}{2\alpha}} + k^{-1} h^{-1}). \quad (28)$$

**Proof:**

We only prove (27). The proof of (28) is similar. Note that

$$\begin{aligned}\hat{u}(t, x) &= \int_0^1 G_\alpha(t, x, y) u_0(y) dy + \int_0^t \int_0^1 G_\alpha(t-s, x, y) d\hat{W}(s, y) \\ &= \int_0^1 G_\alpha(t, x, y) u_0(y) dy + \int_0^t \left[ \int_0^1 G_\alpha(t-s, x, y) \frac{\partial^2 \hat{W}(s, y)}{\partial s \partial y} dy \right] ds,\end{aligned}$$

and

$$\begin{aligned}\hat{u}_t(t, x) &= \int_0^1 \frac{\partial}{\partial t} G_\alpha(t, x, y) u_0(y) dy + \int_0^t \left[ \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) \frac{\partial^2 \hat{W}(s, y)}{\partial s \partial y} dy \right] ds \\ &\quad + \int_0^1 G_\alpha(0, x, y) \frac{\partial^2 \hat{W}(t, y)}{\partial t \partial y} dy.\end{aligned}\tag{29}$$

Since  $w(t, x) = \int_0^1 G_\alpha(t, x, y) w_0(y) dy$  is the solution of the following equation

$$\begin{aligned}\frac{\partial w(t, x)}{\partial t} + (-\Delta)^\alpha w(t, x) &= 0, \quad 0 < x < 1, \quad 0 < t < T, \\ w(t, 0) = w(t, 1) &= 0, \quad 0 < t < T, \\ w(0, x) &= w_0(x),\end{aligned}$$

we therefore have

$$w_0(x) = \int_0^1 G_\alpha(0, x, y) w_0(y) dy.$$

Choose  $w_0(y) = \frac{\partial^2 \hat{W}(t, y)}{\partial t \partial y}$  for fixed  $t$ , we have

$$\int_0^1 G_\alpha(0, x, y) \frac{\partial^2 \hat{W}(t, y)}{\partial t \partial y} dy = \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}.$$

Hence, by (29),

$$\begin{aligned}\hat{u}_t(t, x) &= \int_0^1 \frac{\partial}{\partial t} G_\alpha(t, x, y) u_0(y) dy \\ &\quad + \int_0^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) + \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}.\end{aligned}$$

Using the inequality  $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ ,  $\forall a, b, c \in \mathbb{R}$ , we have

$$\begin{aligned}&\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \hat{u}_t^2(t, x) dx dt \\ &\leq 3\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &\quad + 3\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \frac{\partial \hat{W}(t, x)}{\partial t \partial x} \right]^2 dx dt + 3\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^1 \frac{\partial}{\partial t} G_\alpha(t, x, y) u_0(y) dy \right]^2 dx dt \\ &= 3(I + II + III).\end{aligned}$$

For  $I$ , we have, noting that  $(a+b)^2 \leq 2(a^2+b^2)$ ,  $\forall a, b \in \mathbb{R}$ ,

$$\begin{aligned} I &= 2\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_{j-1}} \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &\quad + 2\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_{t_{j-1}}^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &= 2(I_1 + I_2). \end{aligned}$$

For  $I_1$ , we have, with  $\eta_{kl} = \mathcal{N}(0, 1)$ ,  $k = 0, 1, 2, \dots, J-1$ ,  $l = 0, 1, 2, \dots, j-2$ ,  $j \geq 2$ ,

$$\begin{aligned} I_1 &= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_{j-1}} \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \left[ \sum_{l=0}^{j-2} \int_{t_l}^{t_{l+1}} \sum_{k=0}^{J-1} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s, x, y) \eta_{kl} dy ds \right]^2 dx dt \\ &= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \left[ \sum_{l=0}^{j-2} \sum_{k=0}^{J-1} \left( \int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s, x, y) dy ds \right) \eta_{kl} \right]^2 dx dt \\ &= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{l=0}^{j-2} \sum_{k=0}^{J-1} \left( \int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s, x, y) dy ds \right)^2 dx dt \\ &= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{l=0}^{j-2} \sum_{k=0}^{J-1} \left( \int_{t_l}^{t_{l+1}} \int_{x_k}^{x_{k+1}} \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) dy ds \right)^2 dx dt \\ &= C \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{l=0}^{j-2} \sum_{k=0}^{J-1} \left( \sum_{n=1}^{\infty} \lambda_n^\alpha \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right. \\ &\quad \left. e_n(x) \frac{e^{-\lambda_n^\alpha(t-t_{l+1})} - e^{-\lambda_n^\alpha(t-t_l)}}{\lambda_n^\alpha} \right)^2 dx dt \end{aligned}$$

Note that  $(e_n, e_m) = \delta_{nm}$ ,  $n, m = 1, 2, \dots$ , we have

$$\begin{aligned} I_1 &= C \int_{t_j}^{t_{j+1}} \frac{1}{kh} \sum_{l=0}^{j-2} \sum_{k=0}^{J-1} \sum_{n=1}^{\infty} \left( \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right)^2 \left( \frac{e^{-\lambda_n^\alpha(t-t_{l+1})} - e^{-\lambda_n^\alpha(t-t_l)}}{1} \right)^2 dt \\ &= C \int_{t_j}^{t_{j+1}} \frac{1}{kh} \sum_{l=0}^{j-2} \sum_{k=0}^{J-1} \sum_{n=1}^{\infty} \left( \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right)^2 e^{-2\lambda_n^\alpha(t-t_{l+1})} \left( 1 - e^{-\lambda_n^\alpha(t_{l+1}-t_l)} \right)^2 dt \\ &= C \frac{1}{kh} \sum_{l=0}^{j-2} \sum_{k=0}^{J-1} \sum_{n=1}^{\infty} \left( \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right)^2 \\ &\quad \frac{e^{-2\lambda_n^\alpha(t_j-t_{l+1})} - e^{-2\lambda_n^\alpha(t_{j+1}-t_{l+1})}}{\lambda_n^\alpha} \left( 1 - e^{-\lambda_n^\alpha(t_{l+1}-t_l)} \right)^2 \\ &= C \frac{1}{kh} \sum_{k=0}^{J-1} \sum_{n=1}^{\infty} \left( \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right)^2 \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^\alpha} \sum_{l=0}^{j-2} e^{-2\lambda_n^\alpha(t_j-t_{l+1})} \\ &= C \frac{1}{kh} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} \left( \sum_{k=0}^{J-1} (\cos n\pi x_{k+1} - \cos n\pi x_k)^2 \right) \sum_{l=0}^{j-2} e^{-2\lambda_n^\alpha(t_j-t_{l+1})} \end{aligned}$$

Note that, since  $|\cos(n\pi x_{k+1}) - \cos(n\pi x_k)| \leq (n\pi h)^2$ ,

$$\sum_{k=0}^{J-1} (\cos n\pi x_{k+1} - \cos n\pi x_k)^2 \leq C \sum_{k=0}^{J-1} (n\pi h)^2 = C\lambda_n h.$$

We have, by (68) and (67)

$$\begin{aligned} I_1 &= C \frac{1}{kh} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} (\lambda_n h) \sum_{l=0}^{j-2} e^{-2\lambda_n^\alpha (t_j - t_{l+1})} \\ &= C \frac{1}{kh} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} (\lambda_n h) (k^{-1} \lambda_n^{-\alpha}) \\ &= C k^{-2} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^{2\alpha}} \leq C k^{-2} k^{\frac{4\alpha-1}{2\alpha}} = C k^{-\frac{1}{2\alpha}}. \end{aligned}$$

We remark that  $I_1$  can also be estimated by using the following alternative way.

$$\begin{aligned} I_1 &= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_{j-1}} \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &= \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_{j-1}} \int_0^1 \left( \frac{\partial}{\partial t} G_\alpha(t-s, x, y) \right)^2 dy ds \right] dx dt \\ &= \int_{t_j}^{t_{j+1}} \int_0^1 \int_0^{t_{j-1}} \int_0^1 \left( \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-\lambda_n^\alpha (t-s)} e_n(x) e_n(y) \right)^2 dy ds dx dt \\ &= \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{n=1}^{\infty} \lambda_n^{2\alpha} e^{-2\lambda_n^\alpha (t-s)} ds dt \\ &= \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \lambda_n^{2\alpha} \frac{e^{-2\lambda_n^\alpha (t-t_{j-1})} - e^{-2\lambda_n^\alpha t}}{2\lambda_n^\alpha} dt \end{aligned}$$

Note that  $t \geq t_j$ , we then have, by using (63),

$$I_1 \leq C \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-2\lambda_n^\alpha k} dt = C k \sum_{n=1}^{\infty} \frac{\lambda_n^\alpha}{e^{2\lambda_n^\alpha k}} \leq C k^{-\frac{1}{2\alpha}}.$$

For  $I_2$ , we have

$$\begin{aligned} I_2 &\leq 2\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_{t_{j-1}}^{t_j} \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &\quad + 2\mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_{t_j}^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &= 2I_{21} + 2I_{22}. \end{aligned}$$



For  $I_{21}$ , we have

$$\begin{aligned}
I_{21} &= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \left[ \sum_{k=0}^{J-1} \int_{t_{j-1}}^{t_j} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s, x, y) \eta_{kj} dy ds \right]^2 dx dt \\
&= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \int_{t_{j-1}}^{t_j} \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial t} G_\alpha(t-s, x, y) dy ds \right]^2 dx dt \\
&= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \int_{t_{j-1}}^{t_j} \int_{x_k}^{x_{k+1}} \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) dy ds \right]^2 dx dt \\
&= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \sum_{n=1}^{\infty} \lambda_n^\alpha \frac{e^{-\lambda_n^\alpha(t-t_j)} - e^{-\lambda_n^\alpha(t-t_{j-1})}}{\lambda_n^\alpha} \right. \\
&\quad \left. e_n(x) \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right]^2 dx dt \\
&= \int_{t_j}^{t_{j+1}} \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \sum_{n=1}^{\infty} \left( \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right)^2 e^{-2\lambda_n^\alpha(t-t_j)} (1 - e^{-\lambda_n^\alpha k})^2 \right] dt \\
&= \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \sum_{n=1}^{\infty} \left( \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right)^2 \frac{e^{-2\lambda_n^\alpha(t_j-t_j)} - e^{-2\lambda_n^\alpha(t_{j+1}-t_j)}}{\lambda_n^\alpha} (1 - e^{-\lambda_n^\alpha k})^2 \right] \\
&= \frac{1}{kh} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} \left( \sum_{k=0}^{J-1} (\cos n\pi x_{k+1} - \cos n\pi x_k)^2 \right) \cdot 1 \\
&\leq \frac{1}{kh} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^{\alpha+1}} \lambda_n h = \frac{1}{k} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^\alpha} \leq \frac{1}{k} \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})}{\lambda_n^\alpha}
\end{aligned}$$

Applying (64), we have

$$I_{21} \leq \frac{1}{k} k^{1-\frac{1}{2\alpha}} = k^{-\frac{1}{2\alpha}}.$$

For  $I_{22}$ , we have

$$\begin{aligned}
I_{22} &= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_{t_j}^t \int_0^1 \frac{\partial}{\partial t} G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\
&= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \int_{t_j}^t \int_{x_k}^{x_{k+1}} \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) dy ds \right]^2 dx dt \\
&= \int_{t_j}^{t_{j+1}} \int_0^1 \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \sum_{n=1}^{\infty} \lambda_n^\alpha \frac{e^{-\lambda_n^\alpha(t-t)} - e^{-\lambda_n^\alpha(t-t_j)}}{\lambda_n^\alpha} \right. \\
&\quad \left. e_n(x) \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} dy ds \right]^2 dx dt \\
&= \int_{t_j}^{t_{j+1}} \frac{1}{kh} \sum_{k=0}^{J-1} \left[ \sum_{n=1}^{\infty} \left( \frac{\cos n\pi x_{k+1} - \cos n\pi x_k}{n\pi} \right)^2 (1 - e^{-\lambda_n^\alpha k})^2 \right] dt \\
&= k \frac{1}{kh} \sum_{k=0}^{J-1} \sum_{n=1}^{\infty} \left( \cos n\pi x_{k+1} - \cos n\pi x_k \right)^2 \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n^\alpha}
\end{aligned}$$

Applying (66), we have, noting that  $|\cos(n\pi x_{k+1}) - \cos n\pi x_k| \leq (n\pi h)^2$ ,

$$\begin{aligned} I_{22} &\leq \frac{1}{h} \left( \sum_{k=0}^{J-1} n^2 \pi^2 h^2 \right) \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n^\alpha k})^2}{\lambda_n} \\ &\leq \sum_{n=1}^{\infty} (1 - e^{-\lambda_n^\alpha k})^2 \leq C k^{-\frac{1}{2\alpha}}. \end{aligned} \quad (30)$$

For  $II$ . we have, with  $\eta_{kj} = \mathcal{N}(0, 1)$ ,

$$\begin{aligned} II &= \mathbb{E} \int_{t_j}^{t_{j+1}} \int_0^1 \left( \frac{\partial \hat{W}(t, x)}{\partial t \partial x} \right)^2 dx dt = \mathbb{E} \int_{t_j}^{t_{j+1}} \sum_{k=0}^{J-1} \int_{x_k}^{x_{k+1}} \frac{1}{kh} \eta_{kj}^2 dx dt \\ &= \frac{1}{kh} \int_{t_j}^{t_{j+1}} \sum_{k=0}^{J-1} \int_{x_k}^{x_{k+1}} dx dt = \frac{1}{kh} k = h^{-1}. \end{aligned}$$

Similarly we can estimate  $III$ . Together these estimates complete the proof of Theorem 2.3. ■

### 3. Fourier spectral method

In this section, we will consider Fourier spectral method for solving deterministic space-fractional partial differential equation

$$\frac{\partial \hat{u}(t, x)}{\partial t} + (-\Delta)^\alpha \hat{u}(t, x) = \hat{f}(t, x), \quad 0 < t < T, \quad 0 < x < 1, \quad (31)$$

$$\hat{u}(t, 0) = \hat{u}(t, 1) = 0, \quad 0 < t < T, \quad (32)$$

$$\hat{u}(0, x) = u_0(x), \quad 0 < x < 1, \quad (33)$$

where  $\hat{f}(t, x) = \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}$  is defined by (7) and  $\hat{f} \in L^2((0, T) \times (0, 1))$ .

Denote  $A = -\Delta$  with  $\mathcal{D}(A) = H_0^1(0, 1) \cap H^2(0, 1)$ . For any  $s > 0$  and  $v \in H_0^{2s}(0, 1)$ , we have  $A^s v = \sum_{j=1}^{\infty} \lambda_j^s(v, e_j) e_j$ . It is obvious that

$$|v|_r = \|A^{r/2} v\| = \left( \sum_{j=1}^{\infty} \lambda_j^r(v, e_j)^2 \right)^{1/2}, \quad \forall v \in H_0^r(0, 1), r > 0.$$

Further we denote  $E_\alpha(t) = e^{-tA^\alpha}$ ,  $1/2 < \alpha \leq 1$ . Then the solution of (31)-(33) can be written as the following operator form

$$\hat{u}(t) = E_\alpha(t) \hat{u}_0 + \int_0^t E_\alpha(t-s) \hat{f}(s) ds, \quad \hat{u}_J(0) = u_0. \quad (34)$$

The spectral method of (31)-(33) is to find  $\hat{u}_J(t) \in S_J$  such that

$$\frac{\partial \hat{u}_J(t, x)}{\partial t} + (-\Delta)^\alpha \hat{u}_J(t, x) = P_J \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}, \quad 0 < t < T, \quad 0 < x < 1, \quad (35)$$

$$\hat{u}_J(t, 0) = \hat{u}_J(t, 1) = 0, \quad 0 < t < T, \quad (36)$$

$$\hat{u}_J(0, x) = P_J u_0(x), \quad 0 < x < 1, \quad (37)$$

where  $P_J : H \rightarrow S_J$  is defined by (13).

Similarly the solution of (35)-(37) has the form of

$$\hat{u}_J(t) = E_\alpha(t) P_J \hat{u}_0 + \int_0^t E_\alpha(t-s) P_J \hat{f}(s) ds, \quad \hat{u}_J(0) = P_J u_0. \quad (38)$$

**Theorem 3.1.** Assume that  $\hat{u}$  and  $\hat{u}_J$  are the solutions of (34) and (38), respectively. Let  $0 \leq r < 1/2$  and assume that  $u_0 \in H_0^r(0, 1)$ . Then there exists a positive constant  $C$  such that

$$|\hat{u}(t) - \hat{u}_J(t)|_r \leq C|u_0 - P_J u_0|_r + C \frac{1}{(J+1)^{\alpha(1-r/\alpha)}} \left( \int_0^t \|\hat{f}(s)\|^2 ds \right)^{1/2}. \quad (39)$$

In particular, we have, with  $r = 0$ ,

$$\|\hat{u}(t) - \hat{u}_J(t)\| \leq C\|u_0 - P_J u_0\| + C \frac{1}{(J+1)^\alpha} \left( \int_0^t \|\hat{f}(s)\|^2 ds \right)^{1/2}. \quad (40)$$

To prove Theorem 3.1, we need the following smoothing property for the solution operator  $E_\alpha(t)$ .

**Lemma 3.2.** 1. Let  $s > 0$ . We have, with  $1/2 < \alpha \leq 1$ ,

$$\|A^s E_\alpha(t)\| \leq C t^{-\frac{s}{\alpha}} e^{-\delta t}, \quad t > 0,$$

for some constants  $C$  and  $\delta$  which depend on  $s$  and  $\alpha$ .

2. Let  $P_J : H \rightarrow S_J$  be defined by (13). We have

$$\|E_\alpha(t)(I - P_J)\| \leq e^{-t\lambda_{J+1}^\alpha} \|v\|, \quad t > 0.$$

**Proof:** Note that  $A$  is a positive definite operator with eigenvalues  $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ . For any function  $g(\cdot)$ , we have

$$\|g(A)\| = \sup_{\lambda > \lambda_1 > 0} |g(\lambda)|.$$

Hence, with  $\delta = \frac{1}{2}\lambda_1^\alpha$ ,

$$\begin{aligned} \|A^s E_\alpha(t)\| &= \|A^s E_\alpha(t/2) E_\alpha(t/2)\| \leq \|A^s E_\alpha(t/2)\| \|E_\alpha(t/2)\| \\ &= \sup_{\lambda > \lambda_1} \left( \lambda^s e^{-\frac{t}{2}\lambda^\alpha} \right) \cdot \sup_{\lambda > \lambda_1} \left( e^{-\frac{t}{2}\lambda^\alpha} \right) = \sup_{\lambda > \lambda_1} \left( \frac{\left(\frac{t}{2}\lambda^\alpha\right)^{s/\alpha}}{e^{\frac{t}{2}\lambda^\alpha}} \left(\frac{t}{2}\right)^{-s/\alpha} \right) e^{-\frac{t}{2}\lambda_1^\alpha} \\ &\leq C(t/2)^{-s/\alpha} e^{-\delta t} \leq C t^{-s/\alpha} e^{-\delta t}, \end{aligned}$$

which is (1). To show (2), we note that

$$\|E_\alpha(t)(I - P_J)v\| = \left( \sum_{j=J+1}^{\infty} e^{-2t\lambda_j^\alpha} (v, e_j)^2 \right)^{1/2} \leq e^{-t\lambda_{J+1}^\alpha} \|v\|.$$

The proof of Lemma 3.2 is complete. ■

**Proof of Theorem 3.1:** Subtracting (38) from (34), we get

$$\hat{u}(t) - \hat{u}_J(t) = E_\alpha(t)(u_0 - P_J u_0) + \int_0^t E_\alpha(t-s)(f(s) - P_J f(s)) ds = I + II. \quad (41)$$

For  $I$ , we have, with  $0 \leq r < 1/2$ ,

$$\begin{aligned} |I|_r &= |E_\alpha(t)(u_0 - P_J u_0)|_r = \|A^{\frac{r}{2}} E_\alpha(t)(u_0 - P_J u_0)\| \\ &= \left( \sum_{j=J+1}^{\infty} e^{-2t\lambda_j^\alpha} \lambda_j^r (u_0, e_j)^2 \right)^{1/2} \leq e^{-t\lambda_{J+1}^\alpha} |u_0 - P_J u_0|_r. \end{aligned}$$

For  $II$ , we have, by Lemma 3.2, for some  $\gamma \in (0, 1)$ ,

$$\begin{aligned}
|II|_r &= \left| \int_0^t E_\alpha(t-s)(\hat{f}(s) - P_J \hat{f}(s)) ds \right|_r = \left\| \int_0^t A^{\frac{r}{2}} E_\alpha(t-s)(I - P_J) \hat{f}(s) ds \right\| \\
&\quad \left\| \int_0^t A^{\frac{r}{2}} E_\alpha(1-\gamma)(t-s)E(\gamma(t-s))(I - P_J) \hat{f}(s) ds \right\| \\
&\leq C \int_0^t (t-s)^{-\frac{r}{2\alpha}} e^{-\kappa_\alpha(t-s)} \|\hat{f}(s)\| ds
\end{aligned} \tag{42}$$

where  $\kappa_\alpha = \delta(1-\gamma) + \lambda_{J+1}^\alpha \gamma$ .

By Cauchy- Schwarz inequality, we have

$$|II|_r \leq \left( \int_0^\infty ((t-s)^{-\frac{r}{2\alpha}} e^{-\kappa(t-s)})^2 ds \right)^{1/2} \cdot \left( \int_0^t \|\hat{f}(s)\|^2 ds \right)^{1/2}.$$

Note that  $r < \alpha$ , we have, with  $\lambda_{J+1} = (J+1)^2\pi$ ,

$$\int_0^\infty \frac{e^{-2\kappa_\alpha s}}{s^{r/\alpha}} ds \leq \frac{\int_0^\infty s^{-r/\alpha} e^{-2s} ds}{\kappa_\alpha^{1-r/\alpha}} \leq C \frac{1}{\kappa_\alpha^{1-r/\alpha}} \leq C \frac{1}{(\lambda_{J+1}^\alpha)^{1-r/\alpha}} \leq C \frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}.$$

Thus

$$|II|_r \leq C \frac{1}{(J+1)^{2\alpha(1-r/\alpha)}} \left( \int_0^t \|\hat{f}(s)\|^2 ds \right)^{1/2}.$$

Together these estimates complete the proof of Theorem 3.1.

Combining Theorem 2.2 with Theorem 3.1, we have

**Theorem 3.3.** *Let  $u$  and  $\hat{u}_J$  be the solutions of (18)-(20) and (35)-(37), respectively. Assume that  $u_0 \in H$ . We have, with  $1/2 < \alpha \leq 1$ ,*

$$\begin{aligned}
\mathbb{E} \int_0^T \int_0^1 (u(t, x) - \hat{u}_J(t, x))^2 dx dt &\leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}) + C\|u_0 - P_J u_0\|^2 \\
&\quad + C \frac{1}{(J+1)^{2\alpha}} (k^{-1-\frac{1}{2\alpha}} + k^{-1} h^{-1}).
\end{aligned}$$

**Proof:** Note that

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_0^1 (u(t, x) - \hat{u}_J(t, x))^2 dx dt \\
&\leq 2\mathbb{E} \int_0^T \int_0^1 (u(t, x) - \hat{u}(t, x))^2 dx dt + 2\mathbb{E} \int_0^T \int_0^1 (\hat{u}(t, x) - \hat{u}_J(t, x))^2 dx dt \\
&= 2I + 2II.
\end{aligned}$$

For  $I$ , we have, by Theorem 2.2,

$$I \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}).$$

For  $II$ , we have

$$II = \mathbb{E} \int_0^T \|\hat{u}(t) - \hat{u}_J(t)\|^2 dt \leq C\|u_0 - P_J u_0\|^2 + C \frac{1}{(J+1)^{2\alpha}} \mathbb{E} \int_0^T \int_0^t \|\hat{f}(s)\|^2 ds dt.$$

Note that  $\hat{f}(s) = \hat{u}_s(s) - (-\Delta)^\alpha \hat{u}(s)$ , we have, by Theorem 2.3,

$$\begin{aligned} \mathbb{E} \int_0^T \int_0^t \|\hat{f}(s)\|^2 ds dt &\leq \mathbb{E} \int_0^T \int_0^t \|\hat{u}_s(s) - (-\Delta)^\alpha \hat{u}(s)\|^2 ds dt \\ &\leq C \mathbb{E} \int_0^T \int_0^T \int_0^1 \left( \hat{u}_s^2(s, x) + |(-\Delta)^\alpha \hat{u}(s, x)|^2 \right) dx ds dt \\ &\leq C \sum_{j=0}^N \left( k^{-\frac{1}{2\alpha}} + h^{-1} \right) \leq C \left( k^{-1-\frac{1}{2\alpha}} + k^{-1} h^{-1} \right). \end{aligned}$$

Together these estimates complete the proof of Theorem 3.3. ■

#### 4. Numerical simulations

In this section, we will consider the computational issues for solving the following stochastic space-fractional parabolic partial differential equations by using the spectral method, with  $1/2 < \alpha \leq 1$ ,  $0 < x < 1$ ,  $0 < t \leq T$ , ■

$$\frac{\partial u(t, x)}{\partial t} + \epsilon(-\Delta)^\alpha u(t, x) = f(u(t, x)) + \frac{\partial^2 W(t, x)}{\partial t \partial x}, \quad (43)$$

$$u(t, 0) = u(t, 1) = 0, \quad (44)$$

$$u(0, x) = u_0(x), \quad (45)$$

where  $(-\Delta)^\alpha$  is the fractional Laplacian defined by using the eigenvalues and eigenfunctions of the Laplacian operator  $-\Delta$  subject to some boundary conditions. Here  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function and  $\epsilon > 0$  denotes the diffusion coefficient. In our numerical example, we will use the discrete sine transform MATLAB function **dst** and **idst**. We also include the nonlinear term  $f$  although the error estimates in the previous sections are only proved for  $f = 0$ . In our future work, we will consider the error estimates for solving the nonlinear stochastic space-fractional partial differential equations with multiplicative noise by using the spectral method.

Let  $x_0 < x_1 < \dots < x_J = 1$  be a space partition of  $[0, 1]$  and  $\Delta x = h$  be the space step size. Let  $0 = t_0 < t_1 < \dots < t_N = T$  be the time partition of  $[0, T]$  and  $\Delta t = k$  the time step size. The space-time noise  $\frac{\partial^2 W(t, x)}{\partial t \partial x}$  is approximated by using piecewise constant function  $\frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}$ , where

$$\frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x} = \frac{\eta_{n,j}}{\sqrt{\Delta t \Delta x}}, \quad t_{n-1} \leq t \leq t_n, \quad x_{j-1} \leq x \leq x_j, \quad (46)$$

For convenience, we will denote  $\hat{G}(t, x) = \frac{\partial^2 \hat{W}(t, x)}{\partial t \partial x}$  below.

The equations (43)-(45) can then be approximated by the following, with  $1/2 < \alpha \leq 1$ ,  $0 < x < 1$ ,  $0 < t \leq T$ ,

$$\frac{\partial \hat{u}(t, x)}{\partial t} + \epsilon(-\Delta)^\alpha \hat{u}(t, x) = f(\hat{u}(t, x)) + \hat{G}(t, x), \quad (47)$$

$$\hat{u}(t, 0) = \hat{u}(t, 1) = 0, \quad (48)$$

$$\hat{u}(0, x) = u_0(x). \quad (49)$$

Denote  $A = -\frac{\partial^2}{\partial x^2}$  with  $\mathcal{D}(A) = H_0^1(0, 1) \cap H^2(0, 1)$ . Then  $A$  has eigenvalues  $\lambda_j$  and eigenfunctions  $e_j$  where

$$\lambda_j = j^2 \pi^2, \quad e_j = \sin(j\pi x), \quad j \in \mathbb{Z}^+.$$

That is

$$Ae_j = \lambda_j e_j, \quad j \in \mathbb{Z}^+.$$

The equations (47)-(49) can further be written as the following abstract form: find  $\hat{u}(t) \in H_0^1(0,1) \cap H^2(0,1)$  such that

$$\frac{d\hat{u}(t)}{dt} + A\hat{u}(t) = f(\hat{u}(t)) + \hat{G}(t), \quad 0 < t \leq T \quad (50)$$

$$\hat{u}(0) = u_0, \quad (51)$$

Let  $S_{J-1} := \text{span}\{e_1, e_2, \dots, e_{J-1}\}$ . The spectral method for solving (47) - (49) is to find  $u_{J-1}(t) \in S_{J-1}$  such that, with  $0 < t \leq T$ ,

$$\frac{d\hat{u}_{J-1}(t)}{dt} + A_{J-1}\hat{u}_{J-1}(t) = P_{J-1}f(u_{J-1}(t)) + P_{J-1}\hat{G}(t), \quad (52)$$

$$\hat{u}_{J-1}(0) = P_{J-1}u_0, \quad (53)$$

where  $P_{J-1} : H \rightarrow S_{J-1}$  is the orthogonal projection operator defined by

$$P_{J-1}v = \sum_{j=1}^{J-1} \tilde{v}_j e_j, \quad \tilde{v}_j = (v, e_j),$$

where  $A_{J-1} = P_{J-1}A : S_{J-1} \rightarrow S_{J-1}$  and  $(\cdot, \cdot)$  denotes the inner product in  $H = L^2(0,1)$ . We remark that we use  $S_{J-1}$  (not  $S_J$ ) since we will apply the MATLAB functions **dst** and **idst** in our numerical algorithms below.

The semi-implicit Euler method for solving (47)-(49) is to find  $u_{J-1,n} \approx u_{J-1}(t_n)$  such that

$$\frac{\hat{u}_{J-1,n+1} - \hat{u}_{J-1,n}}{\Delta t} + A_{J-1}\hat{u}_{J-1,n+1} = P_{J-1}f(\hat{u}_{J-1,n}) + P_{J-1}\hat{G}(t_n), \quad (54)$$

$$\hat{u}_{J-1,0} = P_{J-1}u_0, \quad (55)$$

Let

$$\hat{u}_{J-1,n} = \sum_{j=1}^{J-1} \tilde{u}_{j,n} e_j \in S_{J-1}. \quad (56)$$

It is easy to see that the Fourier coefficients  $\tilde{u}_{j,n}$  satisfy, with  $j = 1, 2, \dots, J-1$ ,

$$\tilde{u}_{j,n+1} = (1 + \Delta t \lambda_j)^{-1} \left( \tilde{u}_{j,n} + \Delta t \tilde{f}_j(\hat{u}_{J-1,n}) + \Delta t \tilde{G}_{j,n} \right), \quad (57)$$

$$\tilde{u}_{j,0} = (P_{J-1}u_0, e_j), \quad (58)$$

where

$$P_{J-1}\hat{G}(t_n) = \sum_{j=1}^{J-1} \tilde{G}_{j,n} e_j \in S_{J-1}, \quad P_{J-1}f(\hat{u}_{J-1,n}) = \sum_{j=1}^{J-1} \tilde{f}_j(\hat{u}_{J-1,n}) e_j.$$

Here  $\tilde{u}_{j,n}, \tilde{G}_{j,n}, \tilde{f}_j(\hat{u}_{J-1,n})$  denote the Fourier coefficients of  $\hat{u}_{J-1,n}, \hat{G}(t_n)$  and  $f(\hat{u}_{J-1,n})$ , respectively.

We may use the following steps to describe how to solve (47)-(49) numerically by using the spectral method

**Step 1:** Given initial value  $\hat{u}_0(x)$  and  $f$ , we get the approximation  $u_{J-1,0}(x) = P_{J-1}u_0 \approx u_0$  and  $P_{J-1}f(\hat{u}_{J-1,0}) \approx f(\hat{u}_0(x))$ .

**Step 2:** Find the Fourier coefficients  $\tilde{u}_{j,0}$  and  $\tilde{f}_j(\hat{u}_{J-1,0})$  by

$$\begin{pmatrix} \tilde{u}_{1,0} \\ \tilde{u}_{2,0} \\ \vdots \\ \hat{u}_{J-1,0} \end{pmatrix} = (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_J) \end{pmatrix},$$

$$\begin{pmatrix} \tilde{f}_0(u_{J,0}) \\ \tilde{f}_1(u_{J,0}) \\ \vdots \\ \tilde{f}_J(u_{J,0}) \end{pmatrix} = (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} f(u_0(x_1)) \\ f(u_0(x_2)) \\ \vdots \\ f(u_0(x_J)) \end{pmatrix},$$

and

$$\begin{pmatrix} \tilde{G}_{0,0} \\ \tilde{G}_{1,0} \\ \vdots \\ \tilde{G}_{J-1,0} \end{pmatrix} = (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} \hat{G}(t_0, x_1) \\ \hat{G}(t_0, x_2) \\ \vdots \\ \hat{G}(t_0, x_{J-1}) \end{pmatrix}.$$

Here  $\begin{pmatrix} \hat{G}(t_0, x_1) \\ \hat{G}(t_0, x_2) \\ \vdots \\ \hat{G}(t_0, x_{J-1}) \end{pmatrix} = \hat{W}(1, :)$  and  $\hat{W}$  is generated by

$$\hat{W} = \frac{1}{\sqrt{\Delta t \Delta x}} * \mathbf{randn}(N, J-1). \quad (59)$$

**Step 3:** Find the Fourier coefficients  $\tilde{u}_{j,1}, j = 1, 2, \dots, J$  by

$$\begin{pmatrix} \tilde{u}_{1,1} \\ \tilde{u}_{2,1} \\ \vdots \\ \hat{u}_{J-1,1} \end{pmatrix} = GG ./ EE,$$

where  $./$  denotes the elementwise division and

$$GG = (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} u_0(x_1) \\ u_0(x_2) \\ \vdots \\ u_0(x_J) \end{pmatrix} + \Delta t (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} f(u_0(x_1)) \\ f(u_0(x_2)) \\ \vdots \\ f(u_0(x_J)) \end{pmatrix}$$

$$+ \Delta t (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} \hat{G}(t_0, x_1) \\ \hat{G}(t_0, x_2) \\ \vdots \\ \hat{G}(t_0, x_{J-1}) \end{pmatrix},$$

and, with  $\lambda_j = \pi j$ ,

$$EE = \begin{pmatrix} 1 + \Delta t * \lambda_1^2 \\ 1 + \Delta t * \lambda_2^2 \\ \vdots \\ 1 + \Delta t * \lambda_{J-1}^2 \end{pmatrix}.$$

**Step 4:** Find the Fourier coefficients  $\tilde{u}_{j,2}, j = 1, 2, \dots, J-1$  by

$$\hat{u}_{j,2} = (1 + \Delta t \lambda_j)^{-1} (\tilde{u}_{j,1} + \Delta t \tilde{f}_j(\hat{u}_{J-1,1}) + \Delta t \tilde{G}_{j,1}),$$

where

$$\begin{pmatrix} \tilde{f}_1(\hat{u}_{J-1,1}) \\ \tilde{f}_2(\hat{u}_{J-1,1}) \\ \vdots \\ \tilde{f}_{J-1}(\hat{u}_{J-1,1}) \end{pmatrix} = (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} f(\hat{u}_{J-1,1}(x_1)) \\ f(\hat{u}_{J-1,1}(x_2)) \\ \vdots \\ f(\hat{u}_{J-1,1}(x_{J-1})) \end{pmatrix},$$

and

$$\begin{pmatrix} \tilde{G}_{1,1} \\ \tilde{G}_{2,1} \\ \vdots \\ \tilde{G}_{J-1,1} \end{pmatrix} = (\sqrt{2})^{-1} \left(\frac{J}{2}\right)^{-1} \cdot \mathbf{dst} \begin{pmatrix} \hat{G}(t_1, x_1) \\ \hat{G}(t_1, x_2) \\ \vdots \\ \hat{G}(t_1, x_{J-1}) \end{pmatrix}.$$

Here  $\begin{pmatrix} \hat{G}(t_1, x_1) \\ \hat{G}(t_1, x_2) \\ \vdots \\ \hat{G}(t_1, x_{J-1}) \end{pmatrix} = \hat{W}(2, :)$ , and  $\hat{W}$  is defined in (59).

**Step 5:** Find  $\hat{u}_{J,2}(x_k), k = 1, 2, \dots, J-1$  by

$$\begin{pmatrix} \hat{u}_{J-1,2}(x_1) \\ \hat{u}_{J-1,2}(x_2) \\ \vdots \\ \hat{u}_{J-1,2}(x_{J-1}) \end{pmatrix} = \sqrt{2} \left(\frac{J}{2}\right) \cdot \mathbf{idst} \begin{pmatrix} \tilde{u}_{1,2} \\ \tilde{u}_{2,2} \\ \vdots \\ \tilde{u}_{J-1,2} \end{pmatrix}.$$

**Step 6:** Repeat **Step 3 -5** we obtain all  $\hat{u}_{J-1,n}(x_k), k = 1, 2, \dots, J-1$ .

Let us now introduce the MATLAB functions to solve our problem. Let  $u_0$  denote the initial value vector, that is,  $u_0 = [u_0(x_1), u_0(x_2), \dots, u_0(x_{J-1})]$ . Let  $u$  denote the approximate solution vector at time  $T$ , that is,  $u = [u(x_1, T), u(x_2, T), \dots, u(x_{J-1}, T)]$ . We may use the following MATLAB function to get the approximate solution  $u$  at  $T$  for any function  $f$ . Here we choose  $f(u) = u - u^3$ .

Let  $x = [x_1, x_2, \dots, x_{J-1}]$ ,  $\epsilon = 1$ ,  $\kappa = 1$ . We can obtain the approximate solution  $u$  at time  $T$  at the different  $x_k, k = 1, 2, \dots, J-1$  by the following MATLAB function.

```
function [u]=spde_oned_Gal(u0,x,T,N,kappa,W1,J, epsilon)
dt=T/N; Dt=kappa*dt; % kappa for the different time steps
N=T/Dt;
lambda= pi*[1:(J-1)]'; M= epsilon*lambda.^2; EE=1./(1+Dt*M);
for n=1:N
    u0_hat=(sqrt(2)*J/2)^(-1)*dst(u0);
    f_u0 = u0-u0.^3; % f(u) = u-u^3
    f_u0_hat=(sqrt(2)*J/2)^(-1)*dst(f_u0);
    W=W1(kappa*(n-1)+1,:); W=W'; % kappa for the different tim steps
    G_hat=(sqrt(2)*J/2)^(-1)*dst(W);
    u1_hat=(u0_hat + Dt*f_u0_hat + Dt*G_hat).*EE;
    u1=(sqrt(2)*J/2)*idst(u1_hat);
    u0=u1;
end
u=u1;
```



where  $\mathbf{W1}$  denotes the Brownian sheet generated by

$$W1 = \frac{1}{\sqrt{\Delta t * \Delta x}} * \mathbf{randn}(N, J - 1);$$

**Example 4.1.** Consider, with  $0 < x < 1$ ,  $0 < t \leq T$ , [1], [14],

$$\frac{\partial u(t, x)}{\partial t} + \epsilon(-\Delta)^\alpha u(t, x) = f(u(t, x)) + h(t, x) + \frac{\partial^2 W(t, x)}{\partial t \partial x}, \quad (60)$$

$$u(t, 0) = u(t, 1) = 0, \quad (61)$$

$$u(0, x) = u_0(x), \quad (62)$$

where  $\epsilon = 1$ ,  $f(u) = -bu$ ,  $b = 0.5$  and  $u_0(x) = 10x^2(1 - x)^2$  and

$$h(t, x) = 10(1 + b)x^2(1 - x)^2e^t - 10(2 - 12x + 12x^2)e^t.$$

In Allen, Novosel and Zhang [1] and Du and Zhang [14], they show the numerical approximation of  $\mathbb{E}(u(t, x))$  and  $\mathbb{E}(u(t, x)^2)$  at time  $t = 1$  and  $x = 0.5$  by using finite element method and finite difference method. In Table 1, we obtain the similar approximation values as in their papers for different pair  $(\Delta t, \Delta x)$  by using the spectral method. In our experiment, for each pair  $(\Delta t, \Delta x)$ , 1000 runs are performed. In Table 1,  $U(1, 0.5)$  denotes the approximation of  $u(t, x)$  at  $t = 1$  and  $x = 0.5$ . The computational results converge as  $\Delta t$  and  $\Delta x$  approach to 0.

$\Delta x$	$\Delta t$	$\mathbb{E}U(1, 0.5)$	$\mathbb{E}(U(1, 0.5))^2$
1/4	1/4	1.6108	2.6386
1/4	1/8	1.7003	2.9883
1/4	1/16	1.9051	3.6534
1/4	1/32	1.9051	3.6534
1/8	1/4	1.4838	2.5923
1/8	1/8	1.6574	2.7709
1/8	1/16	1.7323	2.7585
1/8	1/32	1.6676	2.8153
1/16	1/4	1.4681	2.3333
1/16	1/8	1.6097	2.6420
1/16	1/16	1.6110	2.5681
1/16	1/32	1.6133	2.8737
1/32	1/4	1.3605	2.4143
1/32	1/8	1.6099	2.6095
1/32	1/16	1.6839	2.7930
1/32	1/32	1.7061	2.8747

Table 1. The approximation of  $\mathbf{E}u(1, 0.5)$  and  $\mathbf{E}(u(1, 0.5))^2$

In Figure 1, we plot a piecewise constant approximation of the noise  $\hat{G}(t, x)$  with  $J = 2^4$  and  $N = 2^6$  on  $0 \leq t \leq 1$  and  $0 \leq x \leq 1$ .

In Figure 2, we plot an approximation sample path of  $u(t, x)$  with  $J = 2^4$  and  $N = 2^6$  on  $0 \leq t \leq 1$  and  $0 \leq x \leq 1$ .

In Figure 3, we consider the convergence rate against the different time steps. Choose the fixed  $J = 64$ , we then consider the different time steps. The reference solution is obtained by using the time step  $\Delta t_{ref} =$

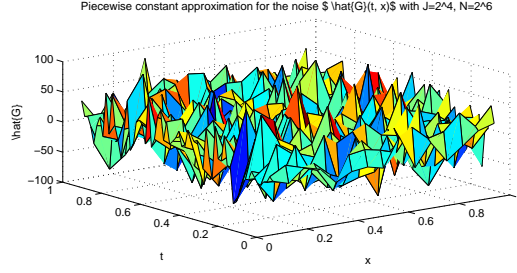


Figure 1: Piecewise constant approximation of the noise  $\hat{G}(t, x)$  with  $J = 2^4$  and  $N = 2^6$

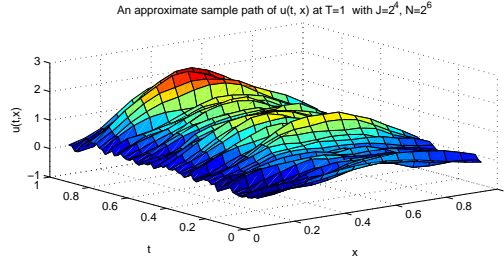


Figure 2: An approximation sample path of  $u(t, x)$  with  $J = 2^4$  and  $N = 2^6$

$T/Nref$  with  $Nref = 10^4$ . Let  $kappa = [20, 50, 100, 150, 200, 250, 300]$ , we will consider the approximate solutions with the different time steps  $\Delta t_i = \Delta tref * kappa(i), i = 1, 2, \dots, 7$ .

In our experiment, for saving the computation time, we will consider the error estimates  $\|\hat{u}_N(t_n) - u(t_n)\|_{L^2(\Omega, H)}$  at time  $t_n$ . We hope to observe the same convergence order as in Theorem 3.3.

To do this, we consider  $M = 100$  simulations. For each simulation  $\omega_m, m = 1, 2, \dots, M$ , we compute  $\hat{u}_N(t_n) \approx \hat{u}(t_n)$  at time  $t_n = 1$  by using the different time steps. We then compute the following  $L^2$  norm of the error at  $t_n = 1$  for the simulations  $\omega_m, m = 1, 2, \dots, M$ ,

$$\epsilon(\Delta t_i, \omega_m) = \epsilon(\Delta t_i, \omega_m, t_n) = \|\hat{u}_N(t_n, \omega_m) - \mathbf{uref}(t_n, \omega_m)\|^2,$$

where the referece ( or “true ”) solution  $\mathbf{uref}(t_n, \omega_m)$  is approximated by the time step  $\Delta tref = T/Nref$ . We then average  $\epsilon(\Delta t_i, \omega_m)$  with respect to  $\omega_m$  to obtain the following approximation of  $\|\hat{u}_N(t_n) - \mathbf{uref}(t_n)\|_{L^2(\Omega, H^r)}$  with respect to the different time step  $\Delta t_i$ ,

$$S(\Delta t_i) = \left( \frac{1}{M} \sum_{m=1}^M \epsilon(\Delta t_i, \omega_m) \right)^{1/2} = \left( \frac{1}{M} \sum_{m=1}^M \|\hat{u}(t_n, \omega_m) - \mathbf{uref}(t_n, \omega_m)\|^2 \right)^{1/2}$$

Since the convergence rate with respect to the time step is  $O(\Delta t^{1/2})$ , i.e.,

$$S(\Delta t_i) \approx \Delta t_i^{1/2},$$

which implies that

$$\log(S(\Delta t_i)) \approx 1/2 \log(\Delta t_i), i = 1, 2, \dots, 7.$$

In Figure 3, we plot the points  $(\log(\Delta t_i), \log(S(\Delta t_i))), i = 1, 2, \dots, 7$  and we see that the points are parallel to the reference line which has the slope  $1/2$  as we expected in our theoretical results.

In Table 2, we list the error  $S(\Delta t_i)$  against the different time steps  $\Delta t_i$ .

$\Delta t_i$	2e-03	5e-03	1e-02	1.5e-02	2e-02	2.5e-02	3e-02
$L^2$ -error	0.2775	0.5355	0.7116	0.9249	1.0306	1.1159	1.1742

Table 2. The  $L^2$  norm error at  $T = 1$  against  $\Delta t$

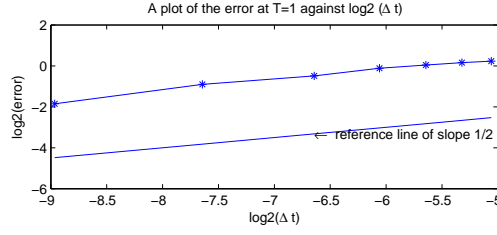


Figure 3: A plot of the error at  $T = 1$  against  $\log_2(\Delta t)$

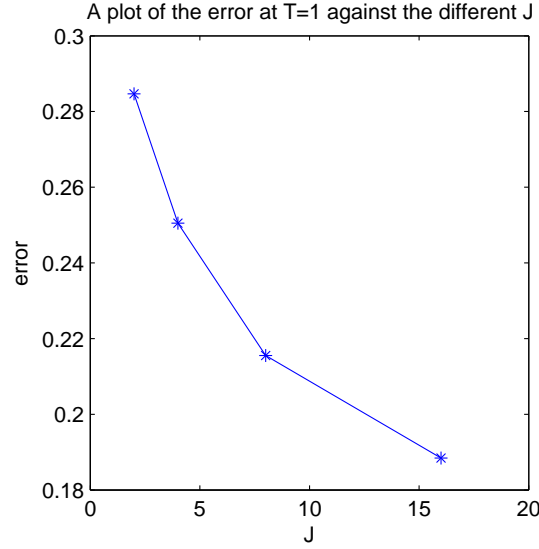


Figure 4: A plot of the error at  $T = 1$  against the  $J$

In Figure 4, we plot the  $L^2$  error  $S(\Delta t)$  against the different  $J$  where the  $L^2$  error are approximated by using  $M = 100$  simulations. We indeed observe the convergence with respect to the different  $J$ .

#### Acknowledgement

We thank Prof. Neville Ford for his consistent support and encouragements for this research. We would also like to thank Dr. Dimitra Antonopoulou and Dr. Nikos Kavallaris for their fruitful discussions about this research topic.

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## 5. Appendix

In this Appendix, we shall provide the proof of Theorem 2.2. To do this, we need the following lemmas.

**Lemma 5.1.** *Let  $1 < \beta \leq 2$ . We have*

$$\sum_{n=1}^{\infty} e^{-n^{\beta}k} n^{\beta} \leq Ck^{-1-\frac{1}{\beta}}, \quad (63)$$

$$\sum_{n=1}^{\infty} \frac{1 - e^{-n^{\beta}k}}{n^{\beta}} \leq Ck^{1-\frac{1}{\beta}}, \quad (64)$$

$$\sum_{n=1}^{\infty} \frac{e^{-n^{\beta}k}}{n^{\beta-2}} \leq Ck^{\frac{\beta-3}{\beta}}, \quad (65)$$

$$\sum_{n=1}^{\infty} (1 - e^{-n^{\beta}k})^2 \leq Ck^{-\frac{1}{\beta}}, \quad (66)$$

$$\sum_{n=1}^{\infty} \frac{(1 - e^{-n^{\beta}k})^2}{n^{2\beta}} \leq Ck^{\frac{2\beta-1}{\beta}}, \quad (67)$$

$$\sum_{l=0}^{j-2} e^{-n^{\beta}(t_j-t_{l+1})} \leq Ck^{-1}n^{-\beta} \quad \text{for } j \geq 2. \quad (68)$$

**Proof:** For (63), we have, with the variable change  $x^{\beta}k = y^{\beta}$ ,

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-n^{\beta}k} n^{\beta} &\leq C \int_1^{\infty} e^{-x^{\beta}k} x^{\beta} dx = C \int_{k^{\frac{1}{\beta}}}^{\infty} e^{-y^{\beta}} (k^{-1}y^{\beta}) k^{-\frac{1}{\beta}} dy \\ &\leq C \int_0^{\infty} e^{-y^{\beta}} k^{-1-\frac{1}{\beta}} y^{\beta} dy \leq Ck^{-1-\frac{1}{\beta}}. \end{aligned}$$

Similarly we can show (64)-(67). For (68), we have, noting that  $1 + x < e^x$ ,  $x > 0$ ,

$$\begin{aligned} \sum_{l=0}^{j-2} e^{-n^{\beta}(t_j-t_{l+1})} &\leq e^{-n^{\beta}k} + (e^{-n^{\beta}k})^2 + \dots \leq e^{-n^{\beta}k} (1 + e^{-n^{\beta}k} + \dots) \\ &\leq e^{-n^{\beta}k} \frac{1}{1 - e^{-n^{\beta}k}} = \frac{1}{e^{n^{\beta}k} - 1} \leq C(n^{\beta}k)^{-1} \leq Ck^{-1}n^{-\beta}. \end{aligned}$$

The proof of the Lemma 5.1 is now complete. ■

We also need the following isometry property for space-time white noise  $W(s, y)$ , see, e.g., [40].

**Lemma 5.2.** *We have*

$$\mathbb{E} \left| \int_0^T \int_0^1 f(s, y) dW(s, y) \right|^2 = \mathbb{E} \int_0^T \int_0^1 f^2(s, y) ds dy.$$

Similarly we have the following isometry property for the approximated space-time white noise  $\hat{W}(s, y)$ , see [1].

**Lemma 5.3.** *We have*

$$\mathbb{E} \left| \int_0^T \int_0^1 f(s, y) d\hat{W}(s, y) \right|^2 = \mathbb{E} \int_0^T \int_0^1 f^2(s, y) ds dy.$$

**Proof:** We have, by (8), Lemma 5.2 and Cauchy-Schwarz inequality,

$$\begin{aligned} & \mathbb{E} \left| \int_0^T \int_0^1 f(s, y) d\hat{W}(s, y) \right|^2 = \mathbb{E} \left| \int_0^T \int_0^1 f(s, y) \frac{\partial^2 \hat{W}(s, y)}{\partial s \partial y} dy ds \right|^2 \\ &= \mathbb{E} \left| \sum_{j=0}^{N-1} \sum_{i=0}^{J-1} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f(s, y) \frac{\int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} dW(r, z)}{kh} dy ds \right|^2 \\ &= \mathbb{E} \left| \sum_{j=0}^{N-1} \sum_{i=0}^{J-1} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \left( \frac{1}{kh} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f(s, y) dy ds \right) dW(r, z) \right|^2 \\ &= \mathbb{E} \sum_{j=0}^{N-1} \sum_{i=0}^{J-1} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \left( \frac{1}{kh} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f(s, y) dy ds \right)^2 dz dr \\ &\leq \mathbb{E} \sum_{j=0}^{N-1} \sum_{i=0}^{J-1} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} \left[ \frac{1}{kh} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f^2(s, y) dy ds \right] dz dr \\ &= \mathbb{E} \sum_{j=0}^{N-1} \sum_{i=0}^{J-1} \int_{t_j}^{t_{j+1}} \int_{x_i}^{x_{i+1}} f^2(s, y) dy ds = \mathbb{E} \int_0^T \int_0^1 f^2(s, y) dy ds. \end{aligned}$$

■

**Proof of Theorem 2.2:** By (21) and (25), we have, noting that  $(a+b+c)^2 \leq 2(a^2+b^2+c^2)$ ,  $\forall a, b, c \in \mathbb{R}$ ,

$$\begin{aligned} & \mathbb{E} \int_0^T \int_0^1 \left( u(t, x) - \hat{u}(t, x) \right)^2 dx dt \\ &= \mathbb{E} \int_0^T \int_0^1 \left[ \int_0^t \int_0^1 G_\alpha(t-s, x, y) dW(s, y) - \int_0^t \int_0^1 G_\alpha(t-s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\ &\leq 3\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left\{ \left[ \int_0^t \int_0^1 G_\alpha(t-s, x, y) dW(s, y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s, x, y) dW(s, y) \right]^2 \right. \\ &\quad + \left[ \int_0^{t_j} \int_0^1 G_\alpha(t_j-s, x, y) dW(s, y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s, x, y) d\hat{W}(s, y) \right]^2 \\ &\quad \left. + \left[ \int_0^{t_j} \int_0^1 G_\alpha(t_j-s, x, y) d\hat{W}(s, y) - \int_0^t \int_0^1 G_\alpha(t_j-s, x, y) d\hat{W}(s, y) \right]^2 \right\} dx dt \\ &= 3(I + II + III). \end{aligned}$$

We first estimate  $II$ . Using the approximation of the space-time white noise (7), we have, by (8),

$$\begin{aligned}
II &= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_j} \int_0^1 G_\alpha(t_j - s, x, y) dW(s, y) \right. \\
&\quad \left. - \int_0^{t_j} \int_0^1 G_\alpha(t_j - s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\
&= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G_\alpha(t_j - r, x, z) dW(r, z) \right. \\
&\quad \left. - \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G_\alpha(t_j - s, x, y) \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} dW(r, z) dy ds \right]^2 dx dt \\
&= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \left( G_\alpha(t_j - r, x, z) \right. \right. \\
&\quad \left. \left. - \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} G_\alpha(t_j - s, x, y) dy ds \right) dW(r, z) \right]^2 dx dt \\
&= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \left( \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \right. \right. \\
&\quad \left. \left. \left( G_\alpha(t_j - r, x, z) - G_\alpha(t_j - s, x, y) \right) dy ds \right) dW(r, z) \right]^2 dx dt
\end{aligned}$$

By isometry property and Cauchy- schwarz inequality, we get

$$\begin{aligned}
II &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \left( \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \right. \\
&\quad \left. \left( G_\alpha(t_j - r, x, z) - G_\alpha(t_j - s, x, y) \right) dy ds \right)^2 dz dr dx dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
&\quad \left( G_\alpha(t_j - r, x, z) - G_\alpha(t_j - s, x, y) \right)^2 dy ds dz dr dx dt
\end{aligned}$$

Further we have

$$\begin{aligned}
II &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} \left( e^{-\lambda_n^\alpha(t_j-r)} e_n(z) - e^{-\lambda_n^\alpha(t_j-s)} e_n(y) \right)^2 dy ds dz dr dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} \left( e^{\lambda_n^\alpha r} e_n(z) - e^{\lambda_n^\alpha s} e_n(y) \right)^2 dy ds dz dr dt \\
&\leq 2 \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{2\lambda_n^\alpha r} dy ds dz dr dt \\
&\quad + 2 \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} e_n^2(y) (e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 dy ds dz dr dt \\
&= 2II_1 + 2II_2.
\end{aligned}$$

For  $II_2$ , we have, noting that  $e_n^2(y) \leq 1$  and  $\sum_{i=0}^{J-1} \int_{x_i}^{x_{i+1}} dx = 1$ ,

$$\begin{aligned}
II_2 &\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 dy ds dz dr dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} \left( e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s} \right)^2 ds dr dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_{t_l}^r \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} \left( e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s} \right)^2 ds \right] dr dt \\
&\quad + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_r^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} \left( e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s} \right)^2 ds \right] dr dt \\
&= II_{21} + II_{22}.
\end{aligned}$$



For  $II_{21}$ , we have

$$\begin{aligned}
II_{21} &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_{t_l}^r \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha(r-s)}\right)^2 ds \right] dr dt \\
&\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_{t_l}^r \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 ds \right] dr dt \\
&\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 ds \right] dr dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 dr dt
\end{aligned}$$

We will show that

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 dr dt \leq Ck^{1-\frac{1}{2\alpha}}. \quad (69)$$

Assume (69) holds at the moment, we then have

$$II_{21} \leq Ck^{1-\frac{1}{2\alpha}}.$$

We now show (69). Note that  $1 - e^{-x} \leq Cx$  for  $x > 0$  and  $1 - e^{-x} \leq 1$  for  $x > 0$ , we have

$$\begin{aligned}
&\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 dr dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 dr dt \\
&\quad + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 dr dt \\
&\leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} (\lambda_n^\alpha k)^2 dr dt \\
&\quad + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \cdot 1^2 dr dt \\
&\leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha k} - e^{-2\lambda_n^\alpha t_j}}{2\lambda_n^\alpha} (\lambda_n^\alpha k)^2 dt + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha} dt \\
&\leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha} (\lambda_n^\alpha k)^2 dt + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha} dt
\end{aligned}$$

Applying (63) and (64), we get

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t_j-r)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 dr dt \leq Ck^{1-\frac{1}{2\alpha}},$$

which is (69).

For  $II_{22}$ , we have

$$\begin{aligned}
II_{22} &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_r^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e^{\lambda_n^\alpha r} - e^{\lambda_n^\alpha s})^2 ds \right] dr dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_r^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j-s)} \left(1 - e^{-\lambda_n^\alpha (s-r)}\right)^2 ds \right] dr dt \\
&\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_r^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j-s)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 ds \right] dr dt \\
&\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \frac{1}{k} \left[ \int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j-s)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 ds \right] dr dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j-s)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 ds dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j-s)} \left(1 - e^{-\lambda_n^\alpha k}\right)^2 ds dt.
\end{aligned}$$

By (69), we get

$$II_{22} \leq Ck^{1-\frac{1}{2\alpha}}.$$

For  $II_1$ , we have

$$\begin{aligned}
II_1 &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \sum_{i=0}^{J-1} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \frac{1}{kh} \int_{t_l}^{t_{l+1}} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{2\lambda_n^\alpha r} dy ds dz dr dt \\
&\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{i=0}^{J-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{2\lambda_n^\alpha r} dy dz dr dt \\
&+ \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} \sum_{i=0}^{J-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}} \\
&\quad \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} (e_n(z) - e_n(y))^2 e^{2\lambda_n^\alpha r} dy dz dr dt
\end{aligned}$$

Noting that  $e_n(z) = \sqrt{2} \sin(n\pi z)$ ,  $|\sin x - \sin y| \leq |x - y|$  and  $|\sin x - \sin y| \leq 2$ , we have

$$\begin{aligned}
II_1 &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{i=0}^{J-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} 2(n\pi h)^2 e^{2\lambda_n^\alpha r} dy dz dr dt \\
&\quad + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} \sum_{i=0}^{J-1} \int_{x_i}^{x_{i+1}} \frac{1}{h} \int_{x_i}^{x_{i+1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha t_j} 8e^{2\lambda_n^\alpha r} dy dz dr dt \\
&\leq C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^{t_{j-1}} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j-r)} (n\pi h)^2 dr dt + C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_{j-1}}^{t_j} \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha (t_j-r)} dr dt \\
&= C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha k} - e^{-2\lambda_n^\alpha t_j}}{\lambda_n^\alpha} (n\pi h)^2 dt + C \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha k}}{\lambda_n^\alpha} dt \\
&\leq C \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha k}}{\lambda_n^{\alpha-1}} h^2 + C \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha k}}{\lambda_n^\alpha}.
\end{aligned}$$

Applying (65) and (64), we get

$$II_1 \leq C(k^{1-\frac{1}{2\alpha}} + h^2 k^{\frac{2\alpha-3}{2\alpha}}).$$

Now we consider  $I$ . We have,

$$\begin{aligned}
I &= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^t \int_0^1 G_\alpha(t-s, x, y) dW(s, y) - \int_0^{t_j} \int_0^1 G_\alpha(t_j-s, x, y) dW(s, y) \right]^2 dx dt \\
&\leq 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_j} \int_0^1 (G_\alpha(t-s, x, y) - G_\alpha(t_j-s, x, y)) dW(s, y) \right]^2 dx dt \\
&\quad + 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_{t_j}^t \int_0^1 G_\alpha(t-s, x, y) dW(s, y) \right]^2 dx dt \\
&\leq 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_j} \int_0^1 (G_\alpha(t-s, x, y) - G_\alpha(t_j-s, x, y)) \right]^2 dy ds dx dt \\
&\quad + 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_{t_j}^t \int_0^1 G_\alpha(t-s, x, y) \right]^2 dy ds dx dt \\
&= 2I_1 + 2I_2.
\end{aligned}$$

For  $I_1$ , we have, by using isometry equality and noting that  $(e_n, e_m) = \delta_{nm}, n, m = 1, 2, \dots$ ,

$$\begin{aligned}
I_1 &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_0^1 \int_0^1 \left( \sum_{n=1}^{\infty} (e^{-\lambda_n^\alpha(t-s)} - e^{-\lambda_n^\alpha(t_j-s)}) e_n(x) e_n(y) \right)^2 dy ds dx dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{n=1}^{\infty} (e^{-\lambda_n^\alpha(t-s)} - e^{-\lambda_n^\alpha(t_j-s)})^2 ds dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t-s)} (1 - e^{-\lambda_n^\alpha(t_j-t)})^2 ds dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha(t-t_j)} - e^{-2\lambda_n^\alpha t}}{2\lambda_n^\alpha} (1 - e^{-\lambda_n^\alpha(t_j-t)})^2 dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n^\alpha(t-t_j)} - e^{-2\lambda_n^\alpha t}}{2\lambda_n^\alpha} e^{-2\lambda_n^\alpha(t_j-t)} (e^{-\lambda_n^\alpha(t-t_j)} - 1)^2 dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha t_j}}{2\lambda_n^\alpha} (e^{-\lambda_n^\alpha(t-t_j)} - 1)^2 dt
\end{aligned}$$

Applying (64), we have, noting that  $1 - e^{-2\lambda_n^\alpha t_j} \leq 1$  and  $1 - e^{-\lambda_n^\alpha k} \leq 1$ ,

$$\begin{aligned}
I_1 &\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1}{2\lambda_n^\alpha} (1 - e^{-\lambda_n^\alpha k})^2 dt \\
&\leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \frac{1}{2\lambda_n^\alpha} (1 - e^{-\lambda_n^\alpha k}) dt \leq Ck^{1-\frac{1}{2\alpha}}.
\end{aligned} \tag{70}$$

For  $I_2$ , we have, by (64) and noting that  $(e_n, e_m) = \delta_{nm}, n, m = 1, 2, \dots$ ,

$$\begin{aligned}
I_2 &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_{t_j}^t \int_0^1 \left[ \sum_{n=1}^{\infty} e^{-\lambda_n^\alpha(t-s)} e_n(x) e_n(y) \right]^2 dy ds dx dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^t \sum_{n=1}^{\infty} e^{-2\lambda_n^\alpha(t-s)} ds dt = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \left( \frac{1 - e^{-2\lambda_n^\alpha(t-t_j)}}{2\lambda_n^\alpha} \right) dt \\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \sum_{n=1}^{\infty} \left( \frac{1 - e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha} \right) dt \leq \sum_{n=1}^{\infty} \frac{1 - e^{-2\lambda_n^\alpha k}}{2\lambda_n^\alpha} \leq Ck^{1-\frac{1}{2\alpha}}.
\end{aligned}$$

Finally we consider  $III$ . We have

$$\begin{aligned}
III &= \mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_j} \int_0^1 (G_\alpha(t_j - s, x, y) - G_\alpha(t - s, x, y)) d\hat{W}(s, y) \right. \\
&\quad \left. + \int_{t_j}^t \int_0^1 G_\alpha(t - s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\
&\leq 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_0^{t_j} \int_0^1 (G_\alpha(t_j - s, x, y) - G_\alpha(t - s, x, y)) d\hat{W}(s, y) \right]^2 \\
&\quad + 2\mathbb{E} \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \left[ \int_{t_j}^t \int_0^1 G_\alpha(t - s, x, y) d\hat{W}(s, y) \right]^2 dx dt \\
&= 2III_1 + 2III_2.
\end{aligned}$$

For  $III_1$ , we have, by isometry property and the estimates for  $I_1$ ,

$$\begin{aligned}
III_1 &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_0^{t_j} \int_0^1 (G_\alpha(t_j - s, x, y) - G_\alpha(t - s, x, y))^2 dy ds dx dt \\
&\leq Ck^{1-\frac{1}{2\alpha}}.
\end{aligned}$$

For  $III_2$ , we have, by isometry property and the estimates for  $I_2$ ,

$$\begin{aligned}
III_2 &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \int_0^1 \int_{t_j}^t \int_0^1 (G_\alpha(t - s, x, y))^2 ds dy dx dt \\
&\leq Ck^{1-\frac{1}{2\alpha}}.
\end{aligned}$$

Together these estimates complete the proof of Theorem 2.2. ■